Question Sheet 9, Differentiation IV. v1 2020-21

More Examples for extra practice

1. Evaluate

$$\lim_{x\to 0}\frac{\sin\left((3+x)^2\right)-\sin 9}{x}.$$

Hint: Recognize it as a derivative.

2. Let $f_n:\mathbb{R}\to\mathbb{R}$ be defined by

$$f_n(x)=\begin{cases} x^n & x>0\\ 0 & x=0\\ -x^n & x<0\end{cases}.$$

By verifying the definition prove that for all $n\geq 1$, the function f_n is $n-1$ times differentiable with $f_n^{(n-1)}$ is continuous on $\mathbb{R}$ but f_n is not n times differentiable.

3. Let $g:\mathbb{R}\to\mathbb{R}$ be defined by

$$g(x)=\begin{cases} x^2\sin\left(\dfrac{\pi}{x^2}\right) & \text{if } x>0,\\ 0 & \text{if } x\leq 0.\end{cases}$$

i) Use the definition to show that g is differentiable at $x=0$ and find the value of $g'(0)$.

ii) Use the Chain Rule for Differentiation to find $g'(x)$ for all $x\neq 0$.

iii) Calculate

$$g'\left(\frac{1}{\sqrt{2n}}\right)$$

where $n\in\mathbb{N}$.

iv) Prove that $\lim_{x\to 0}g'(x)$ does not exist and so g' is not continuous.

Aside: In the previous question a second derivative existed, was continuous but not differentiable. In this question, the first derivative existed but was not continuous. By examining the family of functions $x^k \sin\left(\pi/x^\ell\right)$ for integers k and ℓ you can construct functions that have exactly the number of derivatives you want at a point but then its last derivative is either not continuous at that point or, if continuous, not differentiable.

4. Using the Mean Value Theorem prove that

$$\arcsin x < \frac{x}{\sqrt{1-x^2}}$$

for all $0 < x < 1$.

5. Using the Mean Value Theorem prove that

$$\ln(1+x) > \frac{x}{1+\frac{x}{2}}$$

for $x > 0$.

6. (Exam 2009)

   i) Prove that

   $$2^x = x^2$$

   has at least three real solutions.

   ii) Prove that it has exactly three real solutions.

7. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that if $a > 0$ there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \ln\left(\frac{b}{a}\right) cf'(c)\,,$$

8. i) Prove that

$$\arcsin x + \arccos x$$

is constant on $(-1, 1)$.

What is the value of this constant?

Hint: look at the derivative.

ii) What can you say about

$$\arctan u + \arctan \frac{1}{u}$$

for $u > 0$.

9. Do **not** use L'Hôpital's Rule to evaluate the following limits i-iv, but instead assume the following results:

$$\lim_{x \to 0} \frac{\cos -1}{x^2} = -\frac{1}{2} \quad \text{and} \quad \lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

i)

$$\lim_{x \to 0} \frac{x \cos x - \sin x}{x^3},$$

Hint. Write $x \cos x - \sin x = x \cos x - x + x - \sin x$.

ii)

$$\lim_{x \to 0} \frac{\tan x - x}{x^3},$$

iii)

$$\lim_{x \to 0} \frac{\tan x - x}{\tan^3 x},$$

iv)

$$\lim_{x \to 0} \frac{\sin 3x - 3x}{x^3},$$

v)

$$\lim_{x \to 0} \frac{\sin 3x - 3 \sin x}{x^3}.$$

10. In a question on Sheet 7, you were asked to show that

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$.

Write down $f'(x)$ for all $x \in \mathbb{R}$. Calculate $f^{(2)}(0)$.

Hint you may recall that $\lim_{x\to 0} (\sin x - x)/x^3 = -1/6$.

11. Use the Composition Rule for Differentiation to prove

i)

$$\frac{d}{dy} \arcsin\left(\frac{1}{\cosh y}\right) = -\frac{1}{\cosh y}$$

for $y > 0$.

ii)

$$\frac{d}{dy} (\arctan(\sinh y)) = \frac{1}{\cosh y}$$

for $y \in \mathbb{R}$.

iii) Can you make up an example for arccos with an appropriate hyperbolic function?

12. i) Calculate the first six Taylor Polynomials

$$T_{n,0} (\ln(1+x))\big|_{x=1}, \quad 0 \le n \le 5.$$

Calculate the first 6 approximations to $\ln 2$, using these polynomials *with an appropriate choice of* x.

ii) Give the Taylor Series for $\ln(1-x)$ and

$$\ln\left(\frac{1+x}{1-x}\right)$$

about 0, along with their intervals of convergence.

Note: The series for $\ln((1+x)/(1-x))$ is due to Gregory, 1668

iii) Calculate the first 6 approximations to $\ln 2$, using the first six Taylor polynomials

$$T_{n,0}\left(\ln\left(1-x\right)\right), 0 \leq n \leq 5,$$

with an appropriate choice of x.

iv) Calculate the first 6 approximations to $\ln 2$, using the first six Taylor polynomials

$$T_{n,0}\left(\ln\left(\frac{1+x}{1-x}\right)\right),$$

$0 \leq n \leq 5$, *with an appropriate choice of* x.

13. What is the maximum *possible* error in using $T_{5,0}f(x)$ to approximate $f(x) = \sin x$ on the interval $[-0.25, 0.25]$?

    What is the *actual* error when using the Taylor polynomial to approximate $\sin(12°)$?

14. Approximate $f(x) = \sqrt[3]{x}$ by the quadratic $T_{2,8}f(x)$.

    How accurate is the approximation when $7 \leq x \leq 9$?

15. Show that the Taylor series for $g(x) = (1+x)^{1/2}$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1-2n) (n!)^2} x^n.$$

Hint You need to show that

$$g^{(n)}(0) = (-1)^{n-1} \frac{(2n)!}{4^n n! (2n-1)}$$

for all $n \geq 1$.

16. Show that

i) the Taylor series for $f(x) = 1/\sqrt{(1+x)}$ around $x = 0$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n (n!)^2} x^n,$$

(**Hint** Try to reuse work you have already done. Note that appears in the solution of Question as $2g^{(1)}(x)$, with $g(x) = \sqrt{1+x}$.)

ii) the Taylor series for $h(x) = 1/\sqrt{(1-x^2)}$ around $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} y^{2n}.$$

(**Hint** Use the fact that if the Taylor series of $f(x)$ is $\sum_{n=0}^{\infty} a_n x^n$ then the Taylor series of $f\left(\alpha x^k\right)$ is $\sum_{n=0}^{\infty} a_n \left(\alpha x^k\right)^n$.)

iii) the Taylor Series for $\arcsin x$ around $x = 0$ is

$$\sum_{\ell=0}^{\infty} \frac{(2\ell)! x^{2\ell+1}}{4^\ell (2\ell+1) (\ell!)^2}.$$

(**Note** I am **not** asking for you to prove that any of these series converge to the given function but you might want to think about how you could do this.)

17. Let $f(x) = \sin x$.

i) Prove that

$$f^{(n)}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \left(\cos\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right)\right)$$

for all $n \geq 1$.

ii) Show that for all $n \geq 1$ both sides of the identity,

$$\cos\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right) = (-1)^{n(n-1)/2} \tag{1}$$

are the same.

Hint: Any n can be written as $n = 4m+r$, where r, the remainder on dividing by 4, takes only the values $r = 0, 1, 2$ or 3. Show that the values of both sides of (4) depend only on r, and so there are only 4 cases to check.

iii) Deduce that the Taylor series for $\sin x$ around $a = \pi/4$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{\sqrt{2}n!} \left(x - \frac{\pi}{4}\right)^n .$$

Prove that this series converges to $\sin x$ for all x.